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ON GENERALIZED EIGHTH ORDER MOCK THETA FUNCTIONS

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Abstract: In this paper we have generalized eighth order mock theta functions, recently introduced by Gordon and MacIntosh involving four independent variables. The idea of generalizing was to have four extra parameters, which on specializing give known functions and thus these results hold for those known functions. We have represented these generalized functions as q-integral. Thus on specializing we have the classical mock theta functions represented as q-integral. The same is true for the multibasic expansion given.

Keywords: q-Hypergeometric Series, Mock Theta functions, Continued Fractions, q-Integrals.

1. Introduction

The last gift to mathematics by Ramanujan was mock theta functions. In his last letter to Hardy [5], Ramanujan introduced 17 functions and called them mock theta functions as they were not theta functions and classified them as 4 functions of third order, 10 functions of fifth order and 3 functions of seventh order though Ramanujan did not say what he meant by "order" of mock theta function. Later Watson [12] introduced 3 more mock theta functions of third order. Gordon and McIntosh [7] gave eight more mock theta functions and called them of eighth order. Andrews and Hickerson [3] said the "order" is connected with combinatorics interpretation. Andrews [1] generalized five third order mock theta functions. Srivastava [11] generalized eighth order mock theta function. Recently Choi [4] also generalized mock theta functions of third, fifth, sixth, seventh and tenth order.

Motivated by Andrews' generalization of five of seven third order mock theta functions and Choi's generalization, we have tried to generalize the eighth order mock theta functions by introducing four independent variables. The advantage is that by specializing the parameters we can have known functions.

In this paper we have represented these generalized functions as q-integral and we have also given the multibasic expansion. Thus we have on specializing the parameters, the classical mock theta functions representation as q-integral and the multibasic expansion for generalized functions reduced to classical mock theta function of eighth order.

2. Definitions and notations

The eighth order mock theta functions of Gordon and Mclntosh [7] are

$$S_0(q)=\sum_{n=0}^{\infty}\frac{q^{n^2}(-q;q^2)_n}{(-q^2;q^2)_n},\quad S_1(q)=\sum_{n=0}^{\infty}\frac{q^{n(n+2)}(-q;q^2)_n}{(-q^2;q^2)_n},$$

$$T_0(q)=\sum_{n=0}^{\infty}\frac{q^{(n+1)(n+2)}(-q^2;q^2)_n}{(-q;q^2)_{n+1}},\quad T_1(q)=\sum_{n=0}^{\infty}\frac{q^{n^2+n}(-q^2;q^2)_n}{(-q;q^2)_{n+1}},$$

$$U_0(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}(-q;q^2)_n}{(-q^4;q^4)_n}, \qquad U_1(q) = \sum_{n=0}^{\infty} \frac{q^{(n+1)^2}(-q;q^2)_n}{(-q^2;q^4)_{n+1}},$$

$$V_0(q) = -1 + 2\sum_{n=0}^{\infty} \frac{q^{n^2}(-q;q^2)_n}{(q;q^2)_n} = -1 + 2\sum_{n=0}^{\infty} \frac{q^{2n^2}(-q^2;q^4)_n}{(q;q^2)_{2n+1}},$$

$$V_1(q) = \sum_{n=0}^{\infty} \frac{q^{(n+1)^2}(-q;q^2)_n}{(q;q^2)_{n+1}} = \sum_{n=0}^{\infty} \frac{q^{2n^2+2n+1}(-q^4;q^4)_n}{(q;q^2)_{2n+2}},$$

where

$$(a;q^k)_n = \prod_{j=1}^{n}(1-aq^{k(j-1)}), (a;q^k)_\infty = \prod_{j=1}^{\infty}(1-aq^{k(j-1)}), \quad \text{and} \quad (a;q^k)_0 = 1.$$

3. Generalized eighth order mock theta functions

The four variable generalization of the eighth order mock theta functions are

$$S_0(t,\alpha,\beta,z;q) = \frac{1}{(t)_\infty}\sum_{n=0}^{\infty} \frac{(t)_n q^{n^2-n+n\beta}(-zq;q^2)_n\alpha^n}{(-\alpha q^2;q^2)_n},$$

$$T_0(t,\alpha,\beta,z;q) = \frac{1}{(t)_\infty}\sum_{n=0}^{\infty} \frac{(t)_n q^{n^2+3n+2-n+n\beta}(-q^2/\alpha;q^2)_n}{(-q/z;q^2)_{n+1}z^{n+1}},$$

$$U_0(t,\alpha,\beta,z;q) = \frac{1}{(t)_\infty}\sum_{n=0}^{\infty} \frac{(t)_n q^{n^2-n+n\beta}(-zq;q^2)_n\alpha^n}{(-\alpha q^4;q^4)_n},$$

$$V_0(t,\alpha,\beta,z;q) = -1 + \frac{1}{(t)_\infty}\sum_{n=0}^{\infty} \frac{(t)_n q^{n^2-n+n\beta}(-zq;q^2)_n\alpha^n}{(\alpha q;q^2)_n},$$

$$S_1(t,\alpha,\beta,z;q) = \frac{1}{(t)_\infty}\sum_{n=0}^{\infty} \frac{(t)_n q^{n^2+2n-n+n\beta}(-zq;q^2)_n\alpha^n}{(-\alpha q^2;q^2)_n},$$

$$T_1(t,\alpha,\beta,z;q) = \frac{1}{(t)_\infty}\sum_{n=0}^{\infty} \frac{(t)_n q^{n^2+n-n+n\beta}(-q^2/\alpha;q^2)_n}{(-q/z;q^2)_{n+1}z^{n+1}},$$

$$U_1(t,\alpha,\beta,z;q) = \frac{1}{(t)_\infty}\sum_{n=0}^{\infty} \frac{(t)_n q^{(n+1)^2-n+n\beta}(-zq;q^2)_n}{(-\alpha q^2;q^4)_{n+1}},$$

$$V_1(t,\alpha,\beta,z;q) = \frac{1}{(t)_\infty}\sum_{n=0}^{\infty} \frac{(t)_n q^{(n+1)^2-n+n\beta}(-zq;q^2)_n\alpha^n}{(\alpha q;q^2)_{n+1}}.$$

For $t=0$, $\alpha=1$, $\beta=1$ and $z=1$ these functions reduce to classical mock theta functions.

4. Relation between generalized eighth order mock theta functions

The differential operator D_q [8] is defined as

$$zD_{q,z}F(z,\alpha) = F(z,\alpha) - F(zq,\alpha).$$

By using the differential operator we shall connect the generalized eighth order mock theta functions.

Proposition 1. *The following is true:*

$$\text{(i)}\quad D^2_{q,t}S_0(t,\alpha,\beta,z;q)=S_1(t,\alpha,\beta,z;q),$$
$$\text{(ii)}\quad q^2D^2_{q,t}T_1(t,\alpha,\beta,z;q)=T_0(t,z,\alpha,\beta,z;q).$$

P r o o f. Proof of (i):

$$tD_{q,t}S_0(t,\alpha,\beta,z;q)=\frac{1}{(t)_\infty}\sum_{n=0}^{\infty}\frac{(t)_nq^{n^2-n+n\beta}(-zq;q^2)_n\alpha^n}{(-\alpha q^2;q^2)_n}-\frac{1}{(tq)_\infty}\sum_{n=0}^{\infty}\frac{(tq)_nq^{n^2-n+n\beta}(-zq;q^2)_n\alpha^n}{(-\alpha q^2;q^2)_n}$$
$$=\frac{1}{(t)_\infty}\sum_{n=0}^{\infty}\frac{(t)_nq^{n^2-n+n\beta}(-zq;q^2)_n\alpha^n}{(-\alpha q^2;q^2)_n}-\frac{1}{(t)_\infty}\sum_{n=0}^{\infty}\frac{(t)_nq^{n^2-n+n\beta}(-zq;q^2)_n\alpha^n}{(-\alpha q^2;q^2)_n}(1-tq^n)$$
$$=\frac{t}{(t)_\infty}\sum_{n=0}^{\infty}\frac{(t)_nq^{n^2+n\beta}(-zq;q^2)_n\alpha^n}{(-\alpha q^2;q^2)_n}.$$

Similarly

$$D^2_{q,t}S_0(t,\alpha,\beta,z;q)=\frac{1}{(t)_\infty}\sum_{n=0}^{\infty}\frac{(t)_nq^{n^2+n+n\beta}(-zq;q^2)_n\alpha^n}{(-\alpha q^2;q^2)_n}$$
$$=\frac{1}{(t)_\infty}\sum_{n=0}^{\infty}\frac{(t)_nq^{n^2+2n-n+n\beta}(-zq;q^2)_n\alpha^n}{(-\alpha q^2;q^2)_n}=S_1(t,z,\alpha,\beta;q),$$

which proves (i).

Proof of (ii):

$$D_{q,t}T_1(t,\alpha,\beta,z;q)=\frac{1}{(t)_\infty}\sum_{n=0}^{\infty}\frac{q^{n^2+2n-n+n\beta}(-q^2/\alpha;q^2)_n}{(-q/z;q^2)_{n+1}z^{n+1}},$$

and

$$D^2_{q,t}T_1(t,\alpha,\beta,z;q)=\frac{1}{(t)_\infty}\sum_{n=0}^{\infty}\frac{q^{n^2+3n-n+n\beta}(-q^2/\alpha;q^2)_n}{(-q/z;q^2)_{n+1}z^{n+1}},$$
$$q^2D^2_{q,t}T_1(t,\alpha,\beta,z;q)=\frac{1}{(t)_\infty}\sum_{n=0}^{\infty}\frac{q^{n^2+3n+2-n+n\beta}(-q^2/\alpha;q^2)_n}{(-q/z;q^2)_{n+1}z^{n+1}}=T_0(t,\alpha,\beta,z;q),$$

which proves (ii). □

5. q-Integral representation for the generalized eighth order mock theta functions

Thomae and Jackson [6, p. 19] defined q-integral

$$\int_0^1 f(t)d_qt=(1-q)\sum_{n=0}^{\infty}f(q^n)q^n,$$

using limiting case of q-beta integral, we have

$$\frac{1}{(q^x;q)_\infty}=\frac{(1-q)^{-1}}{(q;q)_\infty}\int_0^1 t^{x-1}(tq;q)_\infty d_qt.$$

We now represent these generalized functions as q-integral. By specializing the parameters we have the integral representation for classical mock theta functions.

Theorem 1.

$$\text{(i)}\quad S_0(q^t,\alpha,\beta,z;q)=\frac{(1-q)^{-1}}{(q;q)_\infty}\int_0^1 u^{t-1}(uq;q)_\infty S_0(0,\alpha,pu,z;q)d_qu,$$
$$\text{(ii)}\quad T_0(q^t,\alpha,\beta,z;q)=\frac{(1-q)^{-1}}{(q;q)_\infty}\int_0^1 u^{t-1}(uq;q)_\infty T_0(0,\alpha,pu,z;q)d_qu,$$
$$\text{(iii)}\quad U_0(q^t,\alpha,\beta,z;q)=\frac{(1-q)^{-1}}{(q;q)_\infty}\int_0^1 u^{t-1}(uq;q)_\infty U_0(0,\alpha,pu,z;q)d_qu,$$
$$\text{(iv)}\quad V_0(q^t,\alpha,\beta,z;q)=\frac{(1-q)^{-1}}{(q;q)_\infty}\int_0^1 u^{t-1}(uq;q)_\infty V_0(0,\alpha,pu,z;q)d_qu,$$
$$\text{(v)}\quad S_1(q^t,\alpha,\beta,z;q)=\frac{(1-q)^{-1}}{(q;q)_\infty}\int_0^1 u^{t-1}(uq;q)_\infty S_1(0,\alpha,pu,z;q)d_qu,$$
$$\text{(vi)}\quad T_1(q^t,\alpha,\beta,z;q)=\frac{(1-q)^{-1}}{(q;q)_\infty}\int_0^1 u^{t-1}(uq;q)_\infty T_1(0,\alpha,pu,z;q)d_qu,$$
$$\text{(vii)}\quad U_1(q^t,\alpha,\beta,z;q)=\frac{(1-q)^{-1}}{(q;q)_\infty}\int_0^1 u^{t-1}(uq;q)_\infty U_1(0,\alpha,pu,z;q)d_qu,$$
$$\text{(viii)}\quad V_1(q^t,\alpha,\beta,z;q)=\frac{(1-q)^{-1}}{(q;q)_\infty}\int_0^1 u^{t-1}(uq;q)_\infty V_1(0,\alpha,pu,z;q)d_qu.$$

P r o o f. A detailed proof for $S_0(q^t,\alpha,\beta,z;q)$ is given.The proofs of the other functions are similar, so omitted.

Proof of (i): By definition

$$S_0(t,\alpha,\beta,z;q)=\frac{1}{(t)_\infty}\sum_{n=0}^{\infty}\frac{(t)_n q^{n^2-n+n\beta}(-zq;q^2)_n\alpha^n}{(-\alpha q^2;q^2)_n}.$$

Replacing t by q^t, we have

$$S_0(q^t,\alpha,\beta,z;q)=\frac{1}{(q^t)_\infty}\sum_{n=0}^{\infty}\frac{(q^t)_n q^{n^2-n+n\beta}(-zq;q^2)_n\alpha^n}{(-\alpha q^2;q^2)_n}=\sum_{n=0}^{\infty}\frac{q^{n^2-n+n\beta}(-zq;q^2)_n\alpha^n}{(-\alpha q^2;q^2)_n(q^{n+t};q)_\infty}$$
$$=\sum_{n=0}^{\infty}\frac{q^{n^2-n+n\beta}(-zq;q^2)_n\alpha^n}{(-\alpha q^2;q^2)_n}\frac{(1-q)^{-1}}{(q;q)_\infty}\int_0^1 u^{t-1}(uq;q)_\infty d_qu,$$

but

$$S_0(0,\alpha,\beta,z;q)=\sum_{n=0}^{\infty}\frac{q^{n^2-n+n\beta}(-zq;q^2)_n\alpha^n}{(-\alpha q^2;q^2)_n},$$

putting $q^\beta=p$, we have

$$S_0(0,\alpha,p,z;q)=\sum_{n=0}^{\infty}\frac{q^{n^2-n}(-zq;q^2)_n\alpha^n p^n}{(-\alpha q^2;q^2)_n},$$

$$S_0(q^t,\alpha,\beta,z;q)=\frac{(1-q)^{-1}}{(q;q)_\infty}\int_0^1 u^{t-1}(uq;q)_\infty S_0(0,\alpha,pu,z;q)d_qu,$$

which proves (i).

The proof of all the other functions is similar. Taking $\alpha=1$, $\beta=1$ and $z=1$ we have the integral representation of the classical eighth order mock theta functions. □

6. Multibasic expansion of generalized eighth order mock theta functions

The following bibasic expansion will be used to give multibasic expansion for the generalized functions.

Theorem 2. *The following is true*:

$$\begin{aligned}\sum_{k=0}^{\infty}\frac{(1-ap^kq^k)(1-bp^kq^{-k})(a,b;p)_k(c,a/bc;q)_kq^k}{(1-a)(1-b)(q,aq/b;q)_k(ap/c,bcp;p)_k}\sum_{m=0}^{\infty}\alpha_{m+k}\\ =\sum_{m=0}^{\infty}\frac{(ap,bp;p)_m(cq,aq/bc;q)_mq^m}{(q,aq/b;q)_m(ap/c,bcp;p)_m}\alpha_m.\end{aligned}\tag{6.1}$$

P r o o f. Using the summation formula [6, (3.6.7), p. 71] we have

$$\begin{aligned}\sum_{k=0}^{n}\frac{(1-ap^kq^k)(1-bp^kq^{-k})}{(1-a)(1-b)}\frac{(a,b;p)_k(c,a/bc;q)_k}{(q,aq/b;q)_k(ap/c,bcp;p)_k}q^k\\ =\frac{(ap,bp;p)_n(cq,aq/bc;q)_n}{(q,aq/b;q)_n(ap/c,bcp;p)_n}\end{aligned}$$

and [9, Lemma 10, p. 57],

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}B(k,n)=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}B(k,n+k),$$

therefore we get the statement of the theorem. □

We will consider the following case of Theorem 2.

Case I. Letting $q\to q^3$ and $c\to\infty$ in Theorem 1, we have

$$\begin{aligned}\sum_{k=0}^{\infty}\frac{(1-ap^kq^{3k})(1-bp^kq^{-3k})(a,b;p)_kq^{(3k^2+3k)/2}}{(1-a)(1-b)(q^3,aq^3/b;q^3)_kb^kp^{(k^2+k)/2}}\sum_{m=0}^{\infty}\alpha_{m+k}\\ =\sum_{m=0}^{\infty}\frac{(ap,bp;p)_mq^{(3m^2+3m)/2}}{(q^3,aq^3/b;q^3)_mb^mp^{(m^2+m)/2}}\alpha_m.\end{aligned}\tag{6.2}$$

Theorem 3. *The multibasic hypergeometric expansion of these generalized functions are*:

(i) $$S_0(t,\alpha,1,z;q)=\frac{1}{(t)_\infty}\sum_{k=0}^{\infty}\frac{(1-tq^{4k-1})(1-k^{-2k+2})(t;q)_{k-1}(-zq;q^2)_kq^{k^2}\alpha^k}{(1-q^{k+2})(-\alpha q^2;q^2)_k}\times\phi\left[\begin{smallmatrix}q;-zq^{2k+1};tq^{3k},q^{3k+3}\\ q^{k+3};-\alpha q^{2k+2}:0\end{smallmatrix};q,q^2,q^3;q\alpha\right],$$

(ii) $$T_0(t,\alpha,1,z;q)=\frac{1}{(t)_\infty}\sum_{k=0}^{\infty}\frac{(1-tq^{4k-1})(1-k^{-2k+2})(t;q)_{k-1}(-q^2/\alpha;q^2)_kq^{k^2+3k+2}\alpha^k}{(1-q^{k+2})(-q/z;q^2)_{k+1}z^{k+1}}\times\phi\left[\begin{smallmatrix}q;-q^{2k+2}/\alpha;tq^{3k},q^{3k+3}\\ q^{k+3};-q^{2k+2}/z:0\end{smallmatrix};q,q^2,q^3;q^4\alpha\right],$$

(iii) $$U_0(t,\alpha,1,z;q)=\frac{1}{(t)_\infty}\sum_{k=0}^{\infty}\frac{(1-tq^{4k-1})(1-k^{-2k+2})(t;q)_{k-1}(-zq;q^2)_kq^{k^2}\alpha^kz^{2k}}{(1-q^{k+2})(-\alpha q^4;q^4)_k}\times\phi\left[\begin{smallmatrix}q;-zq^{2k+1};tq^{3k},q^{3k+3};0\\ q^{k+3};0;0;-\alpha q^{4k+4}\end{smallmatrix};q,q^2,q^3,q^4;z^2q\alpha\right],$$

$$\text{(iv)}\quad V_0(t,\alpha,1,z;q) = \frac{-1}{(t)_\infty} + \frac{1}{(t)_\infty}\sum_{k=0}^{\infty}\frac{(1-tq^{4k-1})(1-k^{-2k+2})(t;q)_{k-1}(-zq;q^2)_k q^{k^2}\alpha^k z^{2k}}{(1-q^{k+2})(\alpha q;q^2)_k}$$
$$\times\,\phi\left[{}^{q;-zq^{2k+1};tq^{3k},q^{3k+3}}_{q^{k+3};-\alpha q^{2k+1}:0}\,;q,q^2,q^3;q\alpha z^2\right],$$

$$\text{(v)}\quad S_1(t,\alpha,1,z;q) = \frac{1}{(t)_\infty}\sum_{k=0}^{\infty}\frac{(1-tq^{4k-1})(1-k^{-2k+2})(t;q)_{k-1}(-zq;q^2)_k q^{k^2+2k}\alpha^k}{(1-q^{k+2})(-\alpha q^2;q^2)_k}$$
$$\times\,\phi\left[{}^{q;-zq^{2k+1};tq^{3k},q^{3k+3}}_{q^{k+3};-\alpha q^{2k+2}:0}\,;q,q^2,q^3;q^3\alpha\right],$$

$$\text{(vi)}\quad T_1(t,\alpha,1,z;q) = \frac{1}{(t)_\infty}\sum_{k=0}^{\infty}\frac{(1-tq^{4k-1})(1-k^{-2k+2})(t;q)_{k-1}(-q^2/\alpha;q^2)_k q^{k^2+k}\alpha^k}{(1-q^{k+2})(-q/z;q^2)_{k+1}z^{k+1}}$$
$$\times\,\phi\left[{}^{q;-q^{2k+2}/\alpha;tq^{3k},q^{3k+3}}_{q^{k+3};-q^{2k+3}/z:0}\,;q,q^2,q^3;q^2z^{-1}\alpha\right],$$

$$\text{(vii)}\quad U_1(t,\alpha,1,z;q) = \frac{1}{(t)_\infty}\sum_{k=0}^{\infty}\frac{(1-tq^{4k-1})(1-k^{-2k+2})(t;q)_{k-1}(-zq;q^2)_k q^{(k+1)^2}\alpha^k z^{2k}}{(1-q^{k+2})(-\alpha q^2;q^4)_{k+1}}$$
$$\times\,\phi\left[{}^{q;-zq^{2k+1};tq^{3k},q^{3k+3}}_{q^{k+3};0;0;-\alpha q^{4k+6}:0}\,;q,q^2,q^3,q^4;q^3z^2\alpha\right],$$

$$\text{(viii)}\quad V_1(t,\alpha,1,z;q) = \frac{1}{(t)_\infty}\sum_{k=0}^{\infty}\frac{(1-tq^{4k-1})(1-k^{-2k+2})(t;q)_{k-1}(-zq;q^2)_k q^{(k+1)^2}\alpha^k z^{2k}}{(1-q^{k+2})(\alpha q;q^2)_{k+1}}$$
$$\times\,\phi\left[{}^{q;-zq^{2k+1};tq^{3k},q^{3k+3}}_{q^{k+3};\alpha q^{2k+3}:0}\,;q,q^2,q^3;q^3z^2\alpha\right].$$

P r o o f. We shall give the proof of (i) only, for others we will state the value of parameters. Proof of (i): Taking $a = t/q$, $b = q^2$, $p = q$ and

$$\alpha_m = \frac{(q^3;q^3)_m(t;q^3)_m(-zq;q^2)_m\alpha^m q^m}{(q^3;q)_m(-\alpha q^2;q^2)_m} \quad \text{in} \quad (6.2),$$

we have

$$\begin{aligned}&\sum_{k=0}^{\infty}\frac{(1-tq^{4k-1})(1-q^{-2k+2})(t/q,q^2;q)_k q^{k^2+k}}{(1-t/q)(1-q^2)(q^3,t;q^3)_k q^{2k}}\\ &\times\sum_{m=0}^{\infty}\frac{(t;q^3)_{m+k}(q^3;q^3)_{m+k}(-zq;q^2)_{m+k}\alpha^{m+k}q^{m+k}}{(q^3;q)_{m+k}(-\alpha q^2;q^2)_{m+k}}\\ &=\sum_{m=0}^{\infty}\frac{(t,q^3;q)_m q^{m^2+m}}{(q^3,t;q^3)_m q^{2m}}\frac{(q^3;q^3)_m(t;q^3)_m(-zq;q^2)_m\alpha^m q^m}{(q^3;q)_m(-\alpha q^2;q^2)_m}.\end{aligned}\tag{6.3}$$

The right hand side is equal to

$$\begin{aligned}&\sum_{m=0}^{\infty}\frac{(t,q^3;q)_m q^{m^2+m}}{(q^3,t;q^3)_m q^{2m}}\frac{(q^3;q^3)_m(t;q^3)_m(-zq;q^2)_m\alpha^m q^m}{(q^3;q)_m(-\alpha q^2;q^2)_m}\\ &=\sum_{m=0}^{\infty}\frac{(t;q)_m(-zq;q^2)_m q^{m^2}\alpha^m}{(-\alpha q^2;q^2)_m} = (t)_\infty S_0(t,\alpha,\beta,z;q).\end{aligned}$$

The left hand side of (6.3) is equal to

$$\sum_{k=0}^{\infty}\frac{(1-tq^{4k-1})(1-q^{-2k+2})(t/q,q^2;q)_k q^{k^2+k}}{(1-t/q)(1-q^2)(q^3,t;q^3)_k q^{2k}}$$
$$\times\sum_{m=0}^{\infty}\frac{(t;q^3)_k(tq^3;q^3)_m(q^3;q^3)_k(q^{3k+3};q^3)_m(-zq;q^2)_k(-zq^{2k+1};q^2)_m\alpha^{m+k}q^{m+k}}{(q^3;q)_k(q^{k+3};q)_m(-\alpha q^2;q^2)_k(-\alpha q^{2k+2};q^2)_m}$$
$$=\sum_{k=0}^{\infty}\frac{(1-tq^{4k-1})(1-q^{-2k+2})(t;q)_{k-1}(-zq;q^2)_k q^{k^2}\alpha^k}{(1-q^{k+2})(-\alpha q^2;q^2)_k}$$
$$\times\sum_{m=0}^{\infty}\frac{(tq^3;q^3)_m(q^{3k+3};q^3)_m(-zq^{2k+1};q^2)_m\alpha^m q^m}{(q^{k+3};q)_m(-\alpha q^{2k+2};q^2)_m}$$
$$=\frac{1}{(t)_\infty}\sum_{k=0}^{\infty}\frac{(1-tq^{4k-1})(1-k^{-2k+2})(t;q)_{k-1}(-zq;q^2)_k q^{k^2}\alpha^k}{(1-q^{k+2})(-\alpha q^2;q^2)_k}$$
$$\times\phi\left[\begin{matrix}q;-zq^{2k+1};tq^{3k},q^{3k+3}\\ q^{k+3};-\alpha q^{2k+2}:0\end{matrix};q,q^2,q^3;q\alpha\right]$$

which proves (i).

Proof of (ii): Take $a=t/q$, $b=q^2$, $p=q$ and

$$\alpha_m=\frac{(q^3;q^3)_m(t;q^3)_m(-q^2/\alpha;q^2)_m q^{4m+2}\alpha^m}{(q^3;q)_m(-q/z;q^2)_{m+1}z^{m+1}}\quad\text{in}\quad(6.2).$$

Proof of (iii): Take $a=t/q$, $b=q^2$, $p=q$ and

$$\alpha_m=\frac{(q^3;q^3)_m(t;q^3)_m(-zq;q^2)_m q^m z^{2m}\alpha^m}{(q^3;q)_m(-\alpha q^4;q^4)_m}\quad\text{in}\quad(6.2).$$

Proof of (iv): Take $a=t/q$, $b=q^2$, $p=q$ and

$$\alpha_m=\frac{(q^3;q^3)_m(t;q^3)_m(-zq;q^2)_m q^m z^{2m}\alpha^m}{(q^3;q)_m(\alpha q;q^2)_m}\quad\text{in}\quad(6.2).$$

Proof of (v): Take $a=t/q$, $b=q^2$, $p=q$ and

$$\alpha_m=\frac{(q^3;q^3)_m(t;q^3)_m(-zq;q^2)_m q^{3m}\alpha^m}{(q^3;q)_m(-\alpha q^2;q^2)_m}\quad\text{in}\quad(6.2).$$

Proof of (vi): Take $a=t/q$, $b=q^2$, $p=q$ and

$$\alpha_m=\frac{(q^3;q^3)_m(t;q^3)_m(-q^2/\alpha;q^2)_m q^{2m}\alpha^m}{(q^3;q)_m(-q/z;q^2)_{m+1}z^{m+1}}\quad\text{in}\quad(6.2).$$

Proof of (vii): Take $a=t/q$, $b=q^2$, $p=q$ and

$$\alpha_m=\frac{(q^3;q^3)_m(t;q^3)_m(-zq;q^2)_m q^{3m+1}z^{2m}\alpha^m}{(q^3;q)_m(-\alpha q^2;q^4)_{m+1}}\quad\text{in}\quad(6.2).$$

Proof of (viii): Take $a=t/q$, $b=q^2$, $p=q$ and

$$\alpha_m=\frac{(q^3;q^3)_m(t;q^3)_m(-zq;q^2)_m q^{3m+1}z^{2m}\alpha^m}{(q^3;q)_m(\alpha q;q^2)_{m+1}}\quad\text{in}\quad(6.2).$$

By taking $\alpha=1$, $\beta=1$ and $z=1$ we have multibasic expansion of classical eighth order mock theta functions.

7. Special cases and Ramanujan's cubic continued fraction

Proposition 2. *We have the following special cases*

$$\text{(i)}\quad U_0(0,-1,1,1;q) = \frac{f(-q,-q)}{\psi(-q)},$$
$$\text{(ii)}\quad U_0(0,-1,1,1;-q) = \frac{f(-q^2,-q^2)}{\psi(-q)},$$
$$\text{(iii)}\quad U_0(0,-1,3,-1;-q) = \frac{f(-q,-q^5)}{\psi(-q)},$$
$$\text{(iv)}\quad U_0(0,-1,1,-1;-q) = \frac{f(-q^3,-q^3)}{\psi(-q)}.$$

P r o o f. Proof of (i): By definition we have

$$U_0(t,\alpha,\beta,z;q) = \frac{1}{(t)_\infty}\sum_{n=0}^{\infty}\frac{(t)_n q^{n^2-n+n\beta}(-zq;q^2)_n\alpha^n}{(-\alpha q^4;q^4)_n}, \tag{7.1}$$

put $t=0$, $\alpha=-1$, $\beta=1$ and $z=1$, therefore we have

$$U_0(0,-1,1,1;q) = \sum_{n=0}^{\infty}\frac{(-1)^n q^{n^2}(-q;q^2)_n}{(q^4;q^4)_n}, \tag{7.2}$$

from [10, eq. (A.13), p. 171], we have

$$\frac{f(-q,-q)}{\psi(-q)} = \sum_{n=0}^{\infty}\frac{(-1)^n q^{n^2}(-q;q^2)_n}{(q^4;q^4)_n}, \tag{7.3}$$

by (7.2) and (7.3), we get

$$U_0(0,-1,1,1;q) = \frac{f(-q,-q)}{\psi(-q)},$$

which proves (i).

Proof of (ii): Put $t=0$, $\alpha=-1$, $\beta=1$, $z=1$ and replace $q=-q$ in (7.1), we have

$$U_0(0,-1,1,1;-q) = \sum_{n=0}^{\infty}\frac{q^{n^2}(q;q^2)_n}{(q^4;q^4)_n}, \tag{7.4}$$

from [10, eq. (A. 23), p. 172], we have

$$\frac{f(-q^2,-q^2)}{\psi(-q)} = \sum_{n=0}^{\infty}\frac{q^{n^2}(q;q^2)_n}{(q^4;q^4)_n}, \tag{7.5}$$

by (7.4) and (7.5), we get

$$U_0(0,-1,1,1;-q) = \frac{f(-q^2,-q^2)}{\psi(-q)},$$

which proves (ii).

Proof of (iii): Put $t = 0$, $\alpha = -1$, $\beta = 3$, $z = -1$ and replace $q = -q$ in (7.1), we have

$$U_0(0, -1, 3, -1; -q) = \sum_{n=0}^{\infty} \frac{q^{n^2+2n}(-q; q^2)_n}{(q^4; q^4)_n}, \tag{7.6}$$

from [10, eq. (A. 52), p. 175], we have

$$\frac{f(-q, -q^5)}{\psi(-q)} = \sum_{n=0}^{\infty} \frac{q^{n^2+2n}(-q; q^2)_n}{(q^4; q^4)_n}, \tag{7.7}$$

by (7.6) and (7.7), we get

$$U_0(0, -1, 3, -1; -q) = \frac{f(-q, -q^5)}{\psi(-q)}, \tag{7.8}$$

which proves (iii).

Proof of (iv): Put $t = 0$, $\alpha = -1$, $\beta = 1$, $z = -1$ and replace $q = -q$ in (7.1), we have

$$U_0(0, -1, 1, -1; -q) = \sum_{n=0}^{\infty} \frac{q^{n^2}(-q; q^2)_n}{(q^4; q^4)_n}, \tag{7.9}$$

from [10, eq. (A. 53), p. 175], we have

$$\frac{f(-q^3, -q^3)}{\psi(-q)} = \sum_{n=0}^{\infty} \frac{q^{n^2}(-q; q^2)_n}{(q^4; q^4)_n}, \tag{7.10}$$

by (7.9) and (7.10), we get

$$U_0(0, -1, 1, -1; -q) = \frac{f(-q^3, -q^3)}{\psi(-q)}, \tag{7.11}$$

which proves (iv). □

Remark 1. Dividing (7.8) by (7.11), we have

$$\frac{U_0(0, -1, 3, -1; -q)}{U_0(0, -1, 1, -1; -q)} = \frac{f(-q, -q^5)}{f(-q^3, -q^3)} = 1 + \frac{q + q^2}{1} {}_{+} \frac{q^2 + q^4}{1} {}_{+} \frac{q^3 + q^6}{1} {}_{+} \cdots$$

which is Ramanujan's cubic continued fraction [2, (3.1.6), p. 86].

8. Conclusion

The advantage of the generalization presented in the paper is that by specializing the parameters we can obtain known functions which connects mock theta functions with continued fractions. So the results obtained for mock theta functions are reduced to continued fractions.

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